Peculiarities of the Minimalist Foundation

for Formal Mathematics

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Trimester Program "**Prospects of Formal Mathematics**"

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Abstract of our talk

- **Why**, **How**, **What** about **Minimalist Foundation** (**MF**) **for formal maths**
- **incompatibility** of most constructive foundations with **Classical Predicativity à la Weyl**
- **compatibility** of **MF** with **Classical Predicativity à la Weyl**:

from
- **Equiconsistency** of **MF** with **its classical version**
- **Dedekind and Cauchy real numbers** in **MF** (**MF** +**EM**) **do not form a set**)

- **Open problems**

why developing Constructive Mathematics?

Bishop 

showed that **constructive mathematics** is **viable**

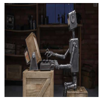
in his book **”Foundations of constructive analysis**

where

proofs are **constructive** when equipped with a **computational contents**

so that

the **existence** of an **object**

can be **computed** by a **machine** 

(like those of **intuitionistic arithmetics** via **Kleene realizability**)

What best foundation for constructive mathematics ??

Since the 80s various foundations for Bishop's constructive mathematics appeared including

in type theory:

Martin-Löf's type theory

Homotopy type theory

in axiomatic set theory:

Aczel's Constructive Zermelo-Fraenkel set theory

. . .

Plurality of foundations $\Longrightarrow$ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory **Homotopy Type Theory** Feferman's constructive expl. maths

the **Minimalist FoundationMF**

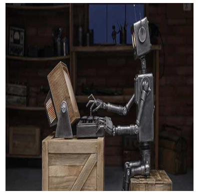
What best foundation for constructive mathematics ??

j.w.w. Giovanni Sambin

To enjoy the plurality of foundations

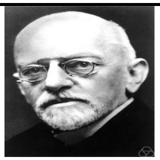
for **Bishop's constructive mathematics** 

look for a a **common core** among them



as a **base** for a **MINIMALIST proof-assistant**

our **foundational** approach

as a **revised Hilbert program**: 

we need of a **trustable** foundation for **mathematics**

compatible with **most relevant foundations**

$\Downarrow$

constructive à la **Bishop** 

open-ended to further extensions according to **Martin-Löf**

for **computed-aided formalization of its proofs** as advocated by **V. Voevodsky**

PREDICATIVE à la **Weyl**

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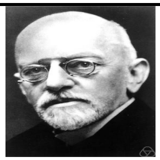
as a **revised Hilbert program**: 

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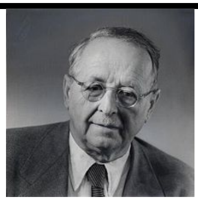
PREDICATIVE à la **Weyl** 

OUR PECULIARITY

classical predicative mathematics is viable

a **former supporter** of classical **predicative mathematics** is

Hermann Weyl 

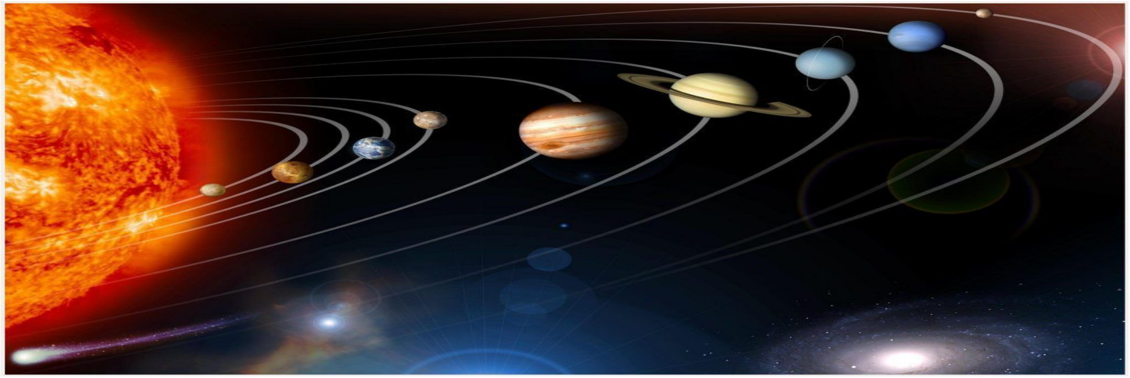
in "**Das Kontinuum**"

advocating that **real numbers do not form a set** in the quote

> " ... **the continuum**... *cannot at all be battered into a single set of elements*.
>
> **Not** *the relationship of an element to a set*,
>
> **but** *of a part to a whole* ought to be taken as a basis for the analysis of a continuum"

modern supporters: " **most basic classical mathematics can be founded predicatively**"

according to **Friedman -Simpson's program**



What known constructive foundations are minimalist ?

i.e. **compatible with most relevant foundations for constructive mathematics?**

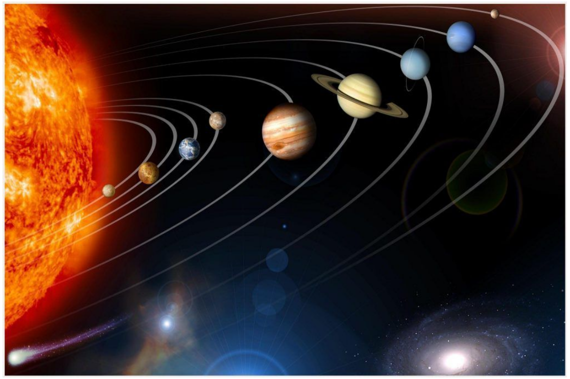
Notion of compatibility between theories

a theory T_1 is compatible with a theory T_2

iff

there is a translation $i\colon T_1 \longrightarrow T_2$

preserving the meaning of logical and set-theoretic operators



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Examples:

Intuitionistic arithmetics is **compatible** with **Classical arithmetics**

Classical arithmetics is **NOT compatible** with **Intuitionistic arithmetics**

Compatibility problem in Bishop's view of constructive mathematics

"A choice function exists in constructive mathematics,

because it is implied by the very meaning of existence"

... due to Brouwer-Heyting-Kolmogorov interpretation of intutionistic logic

but this leads to **incompatiblity** with

the generic internal theory of a topos
Weyl's classical predicativity

Compatibility problems of some relevant foundations

Martin-Löf's type theory (all versions)

NON compatible

with the **internal theory of toposes**

(because of **axiom of choice**)

Aczel's Constructive Zermelo-Fraenkel set theory

and

Homotopy Type Theory

NON compatible

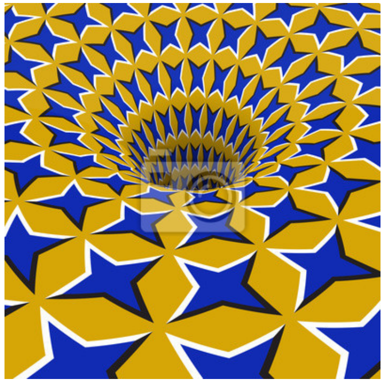
with **classical** **predicativity a' la Weyl**

(because of **exponentiation** of **functional relations**)

Axiom of choice

$$\forall x \in A\, \exists y \in B\, R(x,y) \quad \longrightarrow \quad \exists f \in A \to B\, \forall x \in A\, R(x, f(x))$$

a total relation contains the graph of a function.



An arithmetical theory incompatible with classical predicativity a' la Weyl

Prop.

Heyting arithmetics with finite types $\mathbf{HA}^{\omega}$

+ **internal rule of unique choice $\mathbf{iRC!}_{\mathbf{Nat},\mathbf{Nat}}$** + **excluded middle $\mathbf{EM}$**

is **impredicative**

proof. We can encode the **second order comprehension axiom**

$$(\mathbf{CA})\ \exists \mathbf{f}^{\mathbf{Nat}\ \to\ \mathbf{Nat}}\ \ \forall x^{\mathbf{Nat}}\ (\ f(x) =_{\mathbf{Nat}} 1\ \Leftrightarrow \phi(x)\ )$$

where $\phi(x)$ is an arbitrary formula of the language provided that $\mathbf{f}$ does not occur free in ϕ.

considering variables $\mathbf{f}^{\mathbf{Nat}\ \to\ \mathbf{Nat}}$ as second order variables

The internal rule of unique choice in Heyting arithmetics with finite types

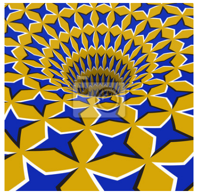
$\forall x^\sigma\ \exists ! y^\tau\ \phi(x, y)$ is derivable in $\mathbf{HA}^\omega$

$\Downarrow$

$\exists f^{\sigma \to \tau}\ \forall x^\sigma\ \phi(x, f x)$ is derivable in $\mathbf{HA}^\omega$, too

i.e. any definable functional relation between natural numbers

contains the graph of a type-theoretic function.



Possible origin of incompatibility with classical predicativity a' la Weyl

the **impredicative arithmetical theory**

$\mathbf{HA}^{\omega} + \mathbf{iRC!}_{\mathbf{Nat},\mathbf{Nat}} +$ **excluded middle EM**

can be interpreted in

Martin-Löf's type theory (all versions) $+\mathbf{EM}$

Homotopy Type Theory $+\mathbf{EM}$ for **h-props**

Aczel's Constructive Zermelo-Fraenkel set theory $+\mathbf{EM}$

Contente-Maietti "On the Compatibility of Constructive Predicative Mathematics with Weyl's Classical Predicativity", **2024**

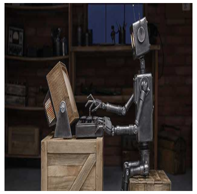
A possible solution to previous compatibility problems

from **Maietti-G.Sambin, "Toward a minimalist foundation for constructive mathematics", 2005**

• perform **program extraction from proofs** in the **metatheory** : i.e. **choice functions exist** only in the **realizability model**
• distinguish **TWO NOTIONS of FUNCTIONS** 1. **functional relations**, **not closed under exponentiation** 2. from **lambda-terms closed under exponentiation**
• make a **two-level foundation** distinguishing **languages** for **extensional math development** / **base for a proof-assistant**

What foundation for COMPUTER-AIDED formalization of proofs?

joint with G. Sambin



a **constructive foundation** should be equipped with

extensional level (used by mathematicians to do their proofs)

$\Downarrow$ interpreted via a **QUOTIENT model**

intensional level (language of computer-aided formalized proofs)

$\Downarrow$

a realizability model (used by computer scientists to extract programs)

Our two-level Minimalist Foundation

from [Maietti "A minimalist two-level foundation for constructive mathematics" Apal 2009]

its **intensional** level is **mTT**

(**minimalist Type Theory**)

= a PREDICATIVE VERSION of Coquand's Calculus of Constructions (Coq).

= first order Martin-Löf's intensional type theory + primitive propositions

+ **one UNIVERSE of small propositions**

its **extensional** level is **emTT**

(**extensional minimalist Type Theory**)

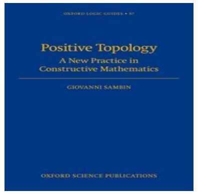
as a PREDICATIVE LOCAL set theory

(NO choice principles)

What mathematics in MF ?

MF was also designed to be a **mathematician user-friendly** foundation

for **Martin-Löf -Sambin**'s **Formal Topology**

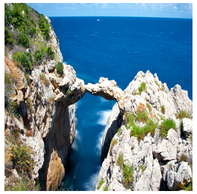
and **Sambin**'s **Positive Topology** in 

by possibly extending MF with **inductive-coinductive** definitions as in

M. Maietti, S. Maschio, M. Rathjen: A realizability semantics for inductive formal topologies, Church's Thesis and Axiom of Choice. **LMCS 2021**

M. Maietti, S. Maschio, M. Rathjen: Inductive and Coinductive Topological Generation with Church's thesis and the Axiom of Choice. **LMCS 2022**

two levels in MF needed for compatibility!



compatibility of our extensional level emTT with extensional theories

Aczel's CZF — Homotopy Type Theory — Internal Th. of topoi — IZF — ZFC

extensional level of MF

compatibility of our intensional level mTT with intensional theories

Martin-Löf's TT — Homotopy Type Theory — Coq

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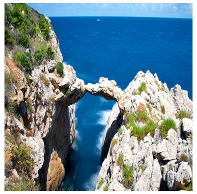

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intensional level of MF

Peculiarity of HoTT

both levels of **MF** are **compatible** with **HoTT**

extensional level of MF

MF-prop as h-prop

HoTT

MF-prop as h-prop

intensional level of MF

M. Contente, M.E. Maietti. **The Compatibility of the Minimalist Foundation with Homotopy Type Theory.** **TCS 2024**

Peculiarity of HoTT: it hosts the whole MF structure!

extensional level of MF

MF-prop as h-prop

HoTT

MF-prop as h-prop

intensional level of MF

MF-prop as set

extraction of programs

Peculiarities of MF

joint with **P. Sabelli**

MF is equiconsistent with MF + excluded middle

$\Rightarrow$ **Dedekind/ Cauchy real numbers** in **MF** (also + **Excluded Middle**) **do not form a set**

$\Rightarrow$ **MF** + **Excluded Middle** is a foundation for **Classical Predicative Maths**

$\Rightarrow$ **MF** is **compatible** with **Classical Predicativity à la Weyl**

Part II. Equiconsistency of MF with its Classical version

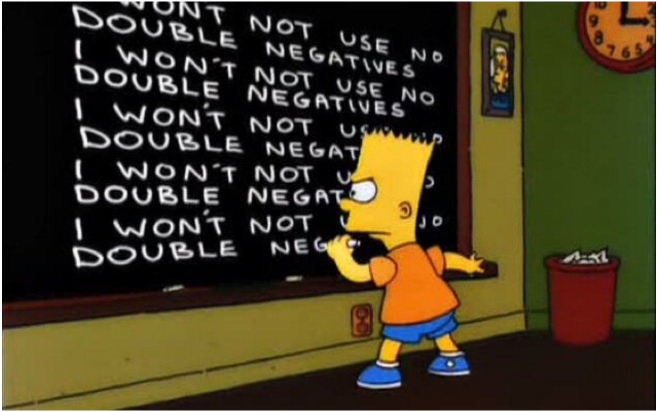

Figure: Simpson chalkboard gag S.11 E.6

The Goal

Using the double-negation translation, Gödel proved the following result.

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A: Yes, it is even equiconsistent with the intuitionistic version.

We proved it by suitably extending Gödel's double-negation translation.

Preliminary Result

Recall the cornerstone of the whole system.

Theorem

The extensional level can be interpreted in the intensional one via a setoid model.[1]

[1] M. E. Maietti. "A minimalist two-level foundation for constructive mathematics". In: *Ann. Pure Appl. Logic* 160.3 (2009), pp. 319–354. ISSN: 0168-0072,1873-2461. DOI: 10.1016/j.apal.2009.01.006. URL: https://doi.org/10.1016/j.apal.2009.01.006.

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As a preliminary result for this work, we proved **its counterpart**.

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The intensional level can be interpreted in the extensional one using canonical isomorphisms.

Since the two levels are equiconsistent, we are justified in **restricting the attention to the extensional level**, which is the one where mathematics is actually developed.

[1] Maietti, "A minimalist two-level foundation for constructive mathematics".

Classical version of Minimalist Foundation

A one-line description of the **extensional level**.

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(Standard) Intuitionistic version

Intuitionistic f.o.l. predicatively typed by $\mathbf{eMLTT} + A/R + \mathcal{P}(A)$.

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In particular, the Minimalist Foundation has primitive propositions among its types (as the Calculus of Constructions).

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Classical version

The same as above, but with classical logic.

The Challenge

Logic and type-theory are highly intertwined:

- terms appear in formulas through equality $a = b$ (as in the one-sorted case);
- types appear in formulas through dependent typing e.g. $\exists x \in A \,.\, \varphi$;
- formulas appear in types as in the quotient set constructor A/R;
- formulas appear in terms as in the subset term constructor $\{x \in A \mid \varphi(x)\} \in \mathcal{P}(A)$.

We need to apply the translation to every entity!

$\neg\neg$-translation for the Minimalist Foundation

Idea: keep translating propositions into $\neg\neg$-stable propositions as in the case of predicate logic, while leaving unaltered type constructors.

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Translation of logic.

$$
\begin{aligned}
\bot^{\mathcal{N}} &:\equiv \bot \\
(\varphi \wedge \psi)^{\mathcal{N}} &:\equiv \varphi^{\mathcal{N}} \wedge \psi^{\mathcal{N}} \\
(\varphi \Rightarrow \psi)^{\mathcal{N}} &:\equiv \varphi^{\mathcal{N}} \Rightarrow \psi^{\mathcal{N}} \\
(\varphi \vee \psi)^{\mathcal{N}} &:\equiv \neg\neg(\varphi^{\mathcal{N}} \vee \psi^{\mathcal{N}}) \\
(\exists x \in A \,.\, \varphi)^{\mathcal{N}} &:\equiv \neg\neg\exists x \in A^{\mathcal{N}} \,.\, \varphi^{\mathcal{N}} \\
(\forall x \in A \,.\, \varphi)^{\mathcal{N}} &:\equiv \forall x \in A^{\mathcal{N}} \,.\, \varphi^{\mathcal{N}} \\
(a =_A b)^{\mathcal{N}} &:\equiv a^{\mathcal{N}} =_{A^{\mathcal{N}}} b^{\mathcal{N}}
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$$

Can you spot the difference with Gödel's translation?

¬¬-translation for the Minimalist Foundation

Idea: keep translating propositions into ¬¬-stable propositions as in the case of predicate logic, while leaving unaltered type constructors.

Translation of type theory.

$$\begin{aligned}
\mathbf{0}^{\mathcal{N}} &:\equiv \mathbf{0} \\
\mathbf{1}^{\mathcal{N}} &:\equiv \mathbf{1} \\
\mathsf{List}(A)^{\mathcal{N}} &:\equiv \mathsf{List}(A^{\mathcal{N}}) \\
(A + B)^{\mathcal{N}} &:\equiv A^{\mathcal{N}} + B^{\mathcal{N}} \\
(\Sigma x \in A \,.\, B)^{\mathcal{N}} &:\equiv \Sigma x \in A^{\mathcal{N}} \,.\, B^{\mathcal{N}} \\
(\Pi x \in A \,.\, B)^{\mathcal{N}} &:\equiv \Pi x \in A^{\mathcal{N}} \,.\, B^{\mathcal{N}} \\
(A/R)^{\mathcal{N}} &:\equiv A^{\mathcal{N}}/R^{\mathcal{N}} \\
\mathcal{P}(A)^{\mathcal{N}} &:\equiv \Sigma U \in \mathcal{P}(A^{\mathcal{N}}) \,.\, (U^{\complement})^{\complement} = U
\end{aligned}$$

$\neg\neg$-translation for the Minimalist Foundation

We say that a type A has $\neg\neg$-*stable equality* if $x =_A y$ is a $\neg\neg$-stable proposition.

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Lemma

All type constructors preserve $\neg\neg$-stable equality.

$\neg\neg$-TRANSLATION FOR THE MINIMALIST FOUNDATION

We say that a type A has $\neg\neg$*-stable equality* if $x =_A y$ is a $\neg\neg$-stable proposition.

LEMMA

All type constructors preserve $\neg\neg$-stable equality.

THEOREM

- *if A is a type, then $A^{\mathcal{N}}$ is a type with $\neg\neg$-stable equality;*
- *if φ is a proposition, then $\varphi^{\mathcal{N}}$ is a $\neg\neg$-stable proposition;*
- *a judgement $\mathcal{J}$ is derivable in the classical version if and only if $\mathcal{J}^{\mathcal{N}}$ is derivable in the intuitionistic version.*

COROLLARY

The Minimalist Foundation is equiconsistent with its classical version.

Application I – Calculus of Constructions

The same techniques and results applies without any substantial changes to the **impredicative version** of the Minimalist Foundation, namely:

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The same techniques and results applies without any substantial changes to the **impredicative version** of the Minimalist Foundation, namely:

Corollary

The extensional version of the Calculus of Constructions is equiconsistent with its extensional, classical version.

Application II – Weyl's Continuum

The real numbers $\mathbb{R}$ can be defined either using Dedekind cuts or Cauchy sequences, and we can prove that they form a *proper* collection, i.e. they cannot be isomorphic to any set.

[2] H. Ishihara and M. E. Maietti and S. Maschio, S. and Streicher, T.

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Proof (Sketch).

Using classical logic we can prove $\mathbb{R} \cong \mathcal{P}(\mathbb{N})$. Thus, if $\mathbb{R}$ were isomorphic to a set, full second-order arithmetic $\mathbf{Z}_2$ could be encoded in the classical version; but this contradicts known proof-theoretic results.[2]Ishihara et al., "Consistency of the intensional level of the Minimalist Foundation with Church's thesis and axiom of choice"

$$\mathbf{Z}_2 \leq \mathbf{MF}_{classical} = \mathbf{MF}_{intuitionistic} \leq \widehat{\mathbf{ID}}_1 < \mathbf{Z}_2$$

$\square$

[2]H. Ishihara and M. E. Maietti and S. Maschio, S. and Streicher, T. Consistency of the intensional level of the Minimalist Foundation with Church's thesis and axiom of choice

Conclusions

- The three levels of the Minimalist Foundation are equiconsistent.
- Contrary to the most relevant foundations for constructive mathematics, the Minimalist Foundation is compatible with classical predicative mathematics.

Open Problems

1. We would like to prove the **equiconsistency of $\mathsf{MF}_{\mathsf{cind}}$** with its classical version. However, $\neg\neg$-translation does not work with (co)inductive constructors!
2. Determine the exact **proof-theoretic strength of MF**.
3. Formalise everything!!

[3] M. E. Maietti and P. Sabelli. "A topological counterpart of well-founded trees in dependent type theory". In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 3 - Proceedings of MFPS XXXIX (Nov. 2023). DOI: 10.46298/entics.11755. URL: https://entics.episciences.org/11755; P. Sabelli. *A topological reading of inductive and coinductive definitions in Dependent Type Theory*. 2024

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First progress

1. We reduced the (co)inductive methods of Formal Topology to common schemes of (co)induction, comparable to Martin-Löf's W/M-types or Aczel's general inductive definitions.[3]
2. Now we can freely interchange between the intensional, extensional, and classical levels.

[3] Maietti and Sabelli, "A topological counterpart of well-founded trees in dependent type theory"; Sabelli, *A topological reading of inductive and coinductive definitions in Dependent Type Theory*.

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Thank you for your attention!