Inductive and Coinductive predicates in the Minimalist Foundation

Pietro Sabelli

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The Minimalist Foundation [9, 5], for short **MF**, is a dependently typed foundational system for constructive mathematics that serves as a common core between the most relevant foundational theories such as Aczel's constructive set theory [1], Martin-Löf's type theory [10], the internal language of a topos [6], the Calculus of Construction [3], or Homotopy Type Theory [11]. Its development has been strongly driven by the desire to find a suitable foundational system for the constructive treatment of topology, known as Formal Topology [13]. In particular, to implement its powerful (co)inductive methods [4], an extension of **MF** with two new constructors formalising the inductive generation of basic covers and the coinductive generation of positivity relations was presented in [8].

In previous joint work with M. E. Maietti presented at CCC2022, we provided a topological counterpart of well-founded sets in terms of inductive suplattices introduced in Martin-Löf-Sambin's formal topology. Here we dualize this result for general (co)inductive predicates. To this purpose, we extend **MF** with context-independent coinductive predicates and we provide a topological counterpart for them in terms of Martin-Löf-Sambin's positivity relations. Furthermore, we show their equivalence with generalized inductive and coinductive definitions of constructive set theory [12] and M-types in extensional Martin-Löf's type theory and Homotopy Type Theory [2]. Our work let us conclude that the extension of **MF** with inductive suplattices and coinductive positivity relations in [8] (see also [7]) has the full strength of the extension of **MF** with general inductive and coinductive definitions.

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