The Internal Language of Quasi-toposes

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The Internal Language of a category

Idea

An *internal language* is a device to turn facts about a category $\mathcal{C}$ into statements of a formal theory $\mathsf{Int}(\mathcal{C})$, so to have a syntax-semantics correspondence.

$$\mathcal{C} \vDash \varphi \Leftrightarrow \mathsf{Int}(\mathcal{C}) \vdash \varphi$$

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Example

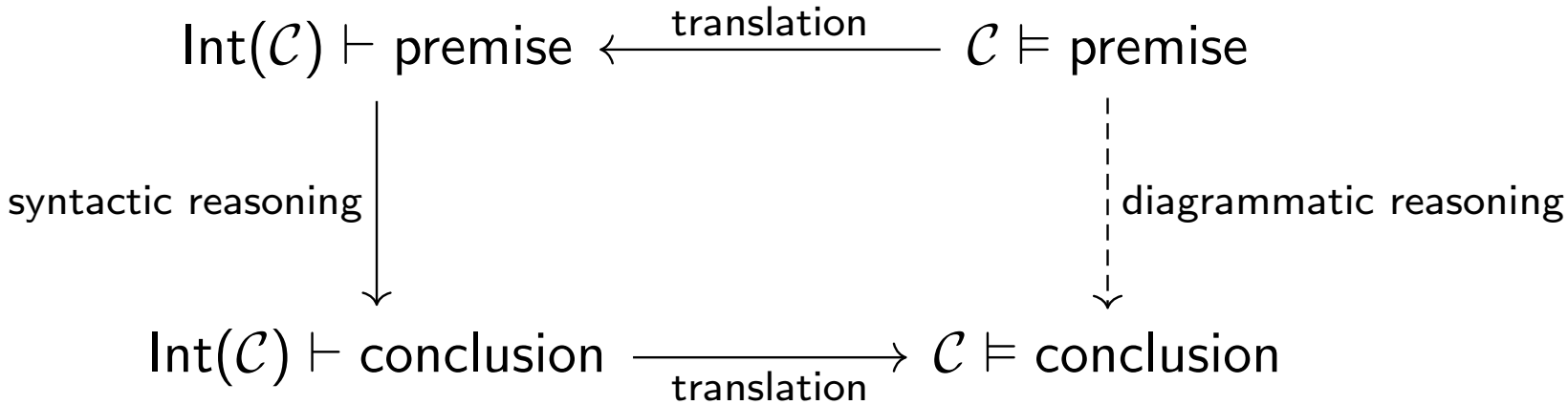
In a topos $\mathcal{E}$, an arrow $f : A \to B$ is an epimorphism whenever the following formula of $\mathsf{Int}(\mathcal{E})$ is derivable.

$$\forall y \in B \; \exists x \in A \,.\, y =_B f(x)$$

The above formula expresses the *surjectivity* of f seen as a function symbol of the syntax.

Internal Language: a powerful tool (cont'd)

The main use case is to avoid complex and unintuitive diagrammatic reasoning.



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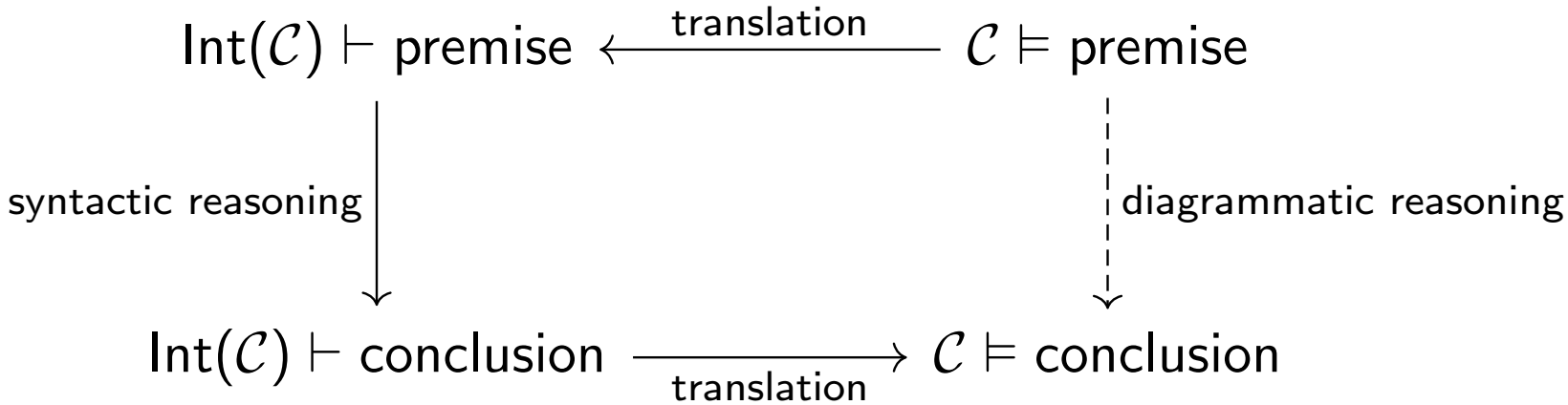

Example

Given a pair of jointly monic arrows $R \rightrightarrows A$ in an arithmetic pretopos, construct its transitive closure $R \rightarrowtail R^{+}$.

The Benabou-Mitchell language for toposes

The internal language of an (arithmetic) topos is given as a theory defined on top of a *simply typed*, *higher-order* calculus.

J. Lambek and P. J. Scott. "Introduction to higher order categorical logic". 1986

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Observation

More of a *many-sorted logic* than a type theory: type constructors are characterized by ad hoc axioms. For example, terms of type $\mathbb{N}$ are governed by Peano's axioms.

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Theorem

There is a pair of functors

$$\mathrm{Synt} : \mathbf{Theory}_{st} \rightleftarrows \mathbf{Topos}_{st} : \mathrm{Int}$$

such that every topos $\mathcal{E}$ *is equivalent to* $\mathrm{Synt}(\mathrm{Int}(\mathcal{E}))$.

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However, it is not the case that every theory is equivalent to the internal language of some topos.

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The Modular Correspondence

It provides *extensional dependently typed* calculi à la Martin-Löf for various categorical structures.

Categories	**Calculus**
Lex	$\mathsf{N}_1, \mathsf{Eq}, \Sigma$
Exact	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \mathsf{Q}$
LCC	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \Pi$
Topos	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \Pi, \Omega$

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Theorem

There is a pair of functors

$$\mathbf{Th} \longleftarrow \mathbf{Th}_{st} \underset{\mathsf{Int}}{\overset{\mathsf{Synt}}{\rightleftarrows}} \mathbf{Top}_{st} \longrightarrow \mathbf{Top}$$

such that $\mathcal{E} \simeq \mathsf{Synt}(\mathsf{Int}(\mathcal{E}))$ *and* $\mathbf{T} \simeq \mathsf{Int}(\mathsf{Synt}(\mathbf{T}))$.

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A Rosetta Stone for Toposes

Calculus	**Topos**
Context, (Closed) Type	Object
Dependent type	Arrow
Dependent mono-type	Monomorphism
Term	Section
Type constructors	*Topos structure*
Singleton set N_1	Terminal object
Dependent sum Σ	Dependent coproduct
Dependent product Π	Dependent product
Equality Eq	Equalizer
Omega type Ω	Subobject classifier

To correctly describe the Omega type Ω, one needs to exploit the correspondence between monomorphisms in the semantics and mono-types (i.e. types with at most one element) in the syntax.

Quasi-toposes

Definition

In a category, a monomorphism is *strong* if it right orthogonal to any epimorphism.

Definition

A *quasi-topos* is a finitely complete, finitely cocomplete, locally cartesian closed category with a *strong*-subobject classifier.

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A *quasi-topos* is a finitely complete, finitely cocomplete, locally cartesian closed category with a *strong*-subobject classifier. Moreover, we say that a quasi-topos is:

- *solid* if the unique map $0 \to 1$ is a strong monomorphism;
- *boolean* if each poset of strong-subobjects is a Boolean algebra;
- *arithmetic* if it has a natural numbers object.

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Example

The category of assemblies **Asm** is an arithmetic, solid, boolean quasi-topos, which is not a topos.

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Taming strong-monomorphisms

Inspired by the design of the Minimalist Foundation, we introduce a primitive kind of type to represent (proof-irrelevant) propositions.

Categories	**Calculus**
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Exact	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \mathsf{Q}$
LCC	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \Pi$
Quasi-topos	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \Pi, \mathsf{N}_0, +, \mathsf{Q}, \mathcal{P}(1)$
Topos	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \Pi, \Omega$

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Observation

Sometimes quasi-toposes are defined with respect to an arbitrary subclass of monomorphisms stable under pullbacks, containing all equalizers, and closed under composition.

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A Rosetta Stone for quasi-toposes

Calculus	**Quasi-topos**
Context, (Closed) Type	Object
Dependent type	Arrow
Dependent mono-type	Monomorphism
Dependent proposition	Strong monomorphism
Term	Section
Type constructors	*Quasitopos structure*
Empty set N_0	Initial object
Singleton set N_1	Terminal object
Dependent sum Σ	Dependent coproduct
Dependent product Π	Dependent product
Disjoint sum $+$	Binary coproduct
Quotient set Q	Coequalizer
Equality Eq	Equalizer
Powerset of the singleton $\mathcal{P}(1)$	Strong-subobject classifier

A calculus for solid and arithmetic quasi-toposes

We can exploit modularity for quasi-toposes too.

Category	**Calculus**
Quasi-topos	$\mathsf{N}_1, \mathsf{Eq}, \Sigma, \Pi, \mathsf{N}_0, +, \mathsf{Q}, \mathcal{P}(1)$
+ solid	+ strong falsum constant
+ boolean	+ Law of the Excluded Middle (LEM)
+ arithmetic	+ $\mathbb{N}$

Recall that the falsum constant can be encoded impredicatively as

$$\bot :\equiv (\forall x \in \mathcal{P}(1))x$$

A falsum constant is *strong* if $\bot \to \mathsf{N}_0$.

Comparison between toposes and quasi-toposes

Proposition

A quasi-topos $\mathcal{E}$ is a topos if and only if one of the following two equivalent conditions is met:

1. (Semantical) every monomorphism of $\mathcal{E}$ is strong;
2. (Logical) the *Rule of Unique Choice* (RC!) holds in $\mathrm{Int}(\mathcal{E})$, i.e. a proof of a statement of the form

$$\forall x \in A \; \exists! y \in B \,.\, \varphi(x, y)$$

always yields an explicit function $f \in A \to B$ such that $\forall x \in A \,.\, \varphi(x, f(x))$.

Applications to Constructive Mathematics

Proposition

The category of assemblies **Asm** *validates the Type-theoretic Church's Thesis*

$$\forall f \in \mathbb{N} \to \mathbb{N}\ \exists e \in \mathbb{N}\,.\, f = \varphi_e \qquad \text{(TCT)}$$

where φ_e is the e-th Turing machine.

M. E. Maietti and P. Sabelli and D. Trotta. "Effectiveness and Continuity in Intuitionistic Quasi-toposes of Assemblies". To appear

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Observation

If **Asm** were a topos, the Rule of Unique Choice would hold in Int(**Asm**). However, already in Heyting Arithmetic with finite types

$$\mathrm{TCT} + \mathrm{LEM} + \mathrm{RC!} \vdash \bot$$

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Applications to Constructive Mathematics (cont'd)

Theorem

Both the Minimalist Foundation and the Calculus of Constructions are consistent with TCT *together with either one of the following:*

1. *classical logic;*
2. *Brouwer's intuitionistic principles.*

Proof.

By interpreting them in (the internal language of) **Asm** formalized in a suitable meta-theory. In particular: for 1. use **ZF** as meta-theory; for 2. use **IZF** + **BP** as meta-theory. □

M. E. Maietti and P. Sabelli. "Equiconsistency of the Minimalist Foundation with its classical version". 2025