A Two-level Foundation for the Calculus of Constructions

Pietro Sabelli
j.w.w. Maria Emilia Maietti

Department of Logic
Institute of Philosophy of the Czech Academy of Sciences

TYPES 2025
University of Strathclyde, Glasgow, 9-13 June 2025

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Full extensional type theories

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To Each His Own: two-level foundations

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3. An interpretation of the extensional level into the intensional one, which reads off an extensional derivation and *restore* its computational content as an intensional judgment.

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Idea

The Minimalist Foundation is a predicative version of the Calculus of Constructions.

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3. The restore interpretation is obtained by lifting the setoid model of the Minimalist Foundation to the present theories.

Equiconsistency Results

Theorem

The two levels $\mathbf{emTT}_{\mathrm{imp}}$ *and* $\mathbf{CC}_{\mathrm{ML}}$ *are equiconsistent.*

M. E. Maietti, P. Sabelli. "Equiconsistency of the Minimalist Foundation with its classical version". 2025

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Let $\mathbf{emTT}^{c}_{\mathrm{imp}}$ be the *classical version* $\mathbf{emTT}_{\mathrm{imp}}$ obtained by adding the Law of Excluded Middle to it.

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Theorem
$\mathbf{emTT}^{c}_{\mathrm{imp}}$ *is equiconsistent with* $\mathbf{emTT}_{\mathrm{imp}}$ *via a double-negation translation.*

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Categorical semantics: quasi-toposes

Definition

A *quasi-topos* is a locally cartesian closed category with finite colimits and a regular subobject classifier.
It is *arithmetical* if moreover has a natural number object.

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Theorem (*)

There is an equivalence of categories

$$\mathbf{ArithQuasiTopos} \quad \underset{\text{Synt}}{\overset{\text{Lang}}{\rightleftarrows}} \quad \mathcal{T}(\mathbf{emTT}_{\text{imp}})$$

$$\mathbf{ArithQuasiTopos} \simeq \mathcal{T}(\mathbf{emTT}_{\text{imp}})$$

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Thanks for your attention!

A Rosetta Stone for quasi-toposes

emTT$_{\text{imp}}$	**Quasi-topos**
Context, (Closed) Type	Object
Dependent type	Arrow
Dependent mono-type	Monomorphism
Dependent proposition	Regular monomorphism
Term	Section
Type constructors	*Quasitopos structure*
Empty set	Initial object
Singleton set	Terminal object
Dependent sum	Dependent coproduct
Dependent product	Dependent product
Disjoint sum	Binary coproduct
Quotient set	Coequalizer
Equality	Equalizer
Universal quantifier	Dependent product
Powerset	Exponentials of the classifier